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CONNECTED ROMAN LINE DOMINATION IN GRAPHS

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ABSTRACT: A connected Roman dominating function $f = (V_0, V_1, V_2)$ on a line graph $L(G) = H$ is a function $f : V(H) \to \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$ such that $\langle V_1 \cup V_2 \rangle$ or $\langle V_2 \rangle$ is connected. The weight of a connected Roman dominating function is the value $f(V(H)) = \sum_{v \in V(H)} f(v)$. The minimum weight of a connected Roman dominating function on a line graph H is called the connected Roman line domination number of G and is denoted by $\gamma_{RCL}(G)$.

In this paper many bounds on $\gamma_{RCL}(G)$ were obtained in terms of elements of G, but not in terms of H. Further we develop its relationship with other different domination parameters of G.

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Introduction: In this paper, we follow the notations of [2] and [3]. All the graphs considered here are simple, finite, non trivial, undirected and connected. As usual $p=|V|$ and $q=|E|$ denote the number of vertices and edges of a graph G respectively.

In general, we use $\langle X\rangle$ to denote the subgraph induced by the set of vertices X and $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v. The notation $\alpha_0(G)$ $(\alpha_1(G))$ is the minimum number of vertex (edge) cover of G. Also $\beta_0(G)$ $(\beta_1(G))$ is the minimum number of vertices (edges) in a maximal independent set of vertex (edge) of G.

A vertex of degree one is called an end vertex and its neighbor is called a support vertex. A vertex v is called a cut vertex if removing it from G increases the number of components of G. A coloring of a graph is an assignment of colors to its vertices so that no two adjacent vertices have the same color. The set of all vertices with any one color is independent and is called a color class. An n-coloring of a graph G uses n colors; it there by partitions V into n color classes. The chromatic number $\chi(G)$ $(\chi'(G))$ is the minimum n for which G has an n vertex (n edge) coloring.

A line graph $L(G)$ is the graph whose vertices correspond to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

Let $G=(V,E)$ be any graph. A set $D\subseteq V$ is said to be dominating set if every vertex not in D is adjacent to some vertex in D. The domination number of G, denoted by $\gamma(G)$ is the minimum cardinality of a minimal dominating set.

A connected Roman dominating function $f=(V_0,V_1,V_2)$ on a line graph $L(G)=H$ is a function $f:V(H)\rightarrow\{0,1,2\}$ satisfying the condition that every vertex u for which $f(u)=0$ is adjacent to at least one vertex v for which $f(v)=2$ such that $\langle V_1\cup V_2\rangle$ or $\langle V_2\rangle$ is connected. The weight of a connected Roman dominating function is the value $f(V(H))=\sum_{v\in V(H)} f(v)$. The minimum weight of a connected Roman dominating function on a line graph H is called the connected Roman line domination number of G and is denoted by

$\gamma_{RCL}(G)$. In this paper many bounds on $\gamma_{RCL}(G)$ were obtained in terms of elements of G, but not in terms of H. Further we develop its relationship with other different domination parameters of G.

We need the following Theorems to our later results.

Theorem A[4]: For any connected graph G, $\gamma(G) \leq \gamma_c(G)$.

Theorem B[5]: For any connected (p,q) graph G with $p \geq 4$ vertices, $\gamma_c(L(G)) \leq p-2$.

Theorem C[2]: For any connected graph G, $\chi \leq \Delta + 1$.

Theorem D[1]: For any graph G, $\kappa \leq \lambda \leq \delta$.

RESULTS

Initially we begin with the connected Roman line domination number of some standard graphs which are straight forward in the following Theorem

Theorem [2]:

i) For any path P_p, with $p \geq 4$ vertices

$\gamma_{RCL}(P_p) = p-2$, if $p = 4$.

$= p-1$, otherwise.

ii) For any cycle C_p, with $p \geq 3$ vertices

$\gamma_{RCL}(C_p) = p-1$, if $p = 3$.

$= p$, otherwise.

iii) For any wheel W_p, with $p \geq 4$ vertices

$\gamma_{RCL}(W_p) = p-1$, if p is odd.

$= p$, if p is even.

iv) For any star $K_{1,n}$, with $n \geq 2$ vertices

$\gamma_{RCL}(K_{1,n}) = 2$.

v) For any complete bipartite graph $K_{m,n}$

$\gamma_{RCL}(K_{m,n}) = m+n$ if $m = n$.

Theorem [2]: For any non trivial (p,q) graph G, $\gamma_{RCL}(G) \leq q$.

Proof: Let G be any non trivial (p,q) graph and $E = \{e_1, e_2, ..., e_q\}$ be the number of edges of G then $V' = \{v_1, v_2, ..., v_q\}$ be the corresponding vertices of $L(G)$ such that $|E| = |V'| = q$. Suppose $\gamma_c' = \{e_1, e_2, ..., e_k\}$ be the number of edges of G which forms γ_c'-set of G. Since every edge of $\{e_k\}$ corresponds to a vertex in $L(G)$. Clearly $\{e_k\}$ forms a connected dominating set $\gamma_c(L(G)) = \{v_1, v_2, ..., v_k\}$ in $L(G)$. Suppose $f = (V_0, V_1, V_2)$ be a γ_{RC}-function in $L(G)$. Then we have $V_0 = V' - v_k$, $V_1 = \{v_i\} \subset v_k$ and $V_2 = v_k - \{v_i\}$. It follows that $\gamma_{RCL}(G) = |v_i \cup v_k| \leq |v_i| + |v_k| = |V'| = q$. Hence $\gamma_{RCL}(G) \leq q$.

Theorem [3]: For any connected (p,q) graph G, $\gamma_{RCL}(G) = q$. If G is isomorphic to P_p with $p \neq 4$ and C_p, $p \geq 4$.

Proof: By Theorem (1), the proof is straight forward.

Theorem [4]: For any non trivial graph G, $\gamma_c(G) \leq \gamma_{RCL}(G)$.

Proof: Let G be any non trivial graph and $f = (V_0, V_1, V_2)$ be a γ_{RC}-function in $L(G)$. Suppose $D_c = \{v_1, v_2, ..., v_n\}$ be the number of vertices forms a γ_c-set in G such that $\gamma_c(G) = |D_c|$ and $D_c' = \{q_1, q_2, ..., q_{n-1}\}$ be the number of edges forms a γ_c'-set in G such that $\gamma_c'(G) = |D_c'|$ and each edge of q_{n-1} is incident with v_{n-1} and v_n where $n \geq 2$. Then we consider the following cases.

Case(1): Suppose $L(G)$ is a trivial graph. Then $V[L(G)] = 1$ and $V(G) = 2$. Hence $\gamma_c(G) = 1 = \gamma_{RCL}(G)$.

Case(2): Suppose $L(G)$ is a non trivial graph. Then $V[L(G)] \geq 2$ and $V(G) \geq 3$ such that $V_2 \neq \phi$. Now consider $\forall$ edge $l \in q_{n-1}; n = 2,3,.....$ which is incident with any two vertices of D_c, then $\exists$ atleast one vertex $m \in V_2$ or $\exists$ the vertices m_1, m_2 such that $m_1 \cup m_2 \in V_1$. Clearly $|D_c| \leq |V_1| + 2|V_2| = \gamma_{RCL}(G)$. Hence $\gamma_c(G) \leq \gamma_{RCL}(G)$.

By the above Theorem we obtain the following result.

Theorem [5]: For any non trivial graph G, $\gamma(G) \leq \gamma_{RCL}(G)$.

Proof: The proof of this Theorem follows from Theorem [A] and Theorem(4). Hence $\gamma(G) \leq \gamma_{RCL}(G)$.

Theorem [6]: For any (p, q) tree T with k end vertices, then $\gamma_{RCL}(T) \geq p - k$.

Proof: Let $F = \{v_1, v_2, ..., v_k\}$ be the number of end vertices of T such that $|F| = k$ and $f = (V_0, V_1, V_2)$ be a γ_{RC}-function in $L(T)$. Suppose $C \subseteq V(T) - F$ be the number of non end vertices of T. Then each block $\{e_n\}$ is incident with the vertices of C gives a complete subgraph in $L(T)$ such that $\forall$ $e = uv \in \{e_n\}$, $(u, v) \notin F$. Suppose $\{e_i; 1 \leq i \leq n\}$ be the number of blocks incident with the end vertices of T. Then $\{e_i\} \in V_0$ and $\{e_n\} \in V_1$ or V_2. If $V_1 = \phi$, then $\langle e_n \rangle$ forms a γ_{RCL}-set in $L(T)$. If $V_1 \neq \phi$, let $\{\{e_j\}, \{e_k\}\} \subseteq \{e_n\}$, then we have $\{e_j\} \in V_1$ and $\{e_k\} \in V_2$

such that $\langle e_j \cup e_k \rangle$ forms a γ_{RCL}-set in $L(T)$, then $|e_j \cup e_k| \ge p-k$, which gives $\gamma_{RCL}(T) \ge p-k$.

Theorem [7]: For any (p,q) graph G, $\gamma_{RCL}(G) \le p+\gamma_c(G)-1$.

Proof: Let G be any (p,q) graph with $V(G) = \{v_1, v_2, ..., v_n\}$ and $f = (V_0, V_1, V_2)$ be a γ_{RC}-function in $L(G)$. Suppose $C = \{v_1, v_2, ..., v_k\} \subseteq V(G)$ be the set of all vertices with $\deg(v) \ge 2$. Then there exists a minimal vertex set $C' \subseteq C$, which covers all the vertices of G. Clearly C' forms a minimal γ-set of G. Suppose the subgraph $\langle C' \rangle$ has only one component. Then C' itself is a connected dominating set of G. Otherwise, if the subgraph $\langle C' \rangle$ has more than one component, then attach the minimum number of vertices $\{v_i\} \in V(G) - C'$, where $\deg(v_i) \ge 2$, which are between the vertices of C' such that $C_1 = C' \cup \{v_i\}$ forms exactly one component in the subgraph $\langle C_1 \rangle$. Clearly, C_1 forms a minimal γ_c-set of G. Let $B = \{e_1, e_2, ..., e_n\}$ be the set of edges which are incident to the vertices of C_1 such that B forms γ_c'-set in G. Now without loss of generality, $B \subseteq V[L(G)]$, let $B_1, B_2 \subseteq B$ such that $B_1 \in V_1$ and $B_2 \in V_2$. If $V_1 \neq \phi$. Then $\langle B_1 \cup B_2 \rangle$ forms a γ_{RCL}-set in $L(G)$. Otherwise $\langle B_2 \rangle$ forms a γ_{RCL}-set in $L(G)$. Clearly $|B_1| + 2|B_2| \le |p| + |C_1| - 1$. Hence $\gamma_{RCL}(G) \le p+\gamma_c(G)-1$.

Theorem [8]: For any non trivial tree T with n blocks, then $\gamma_{RCL}(T) \le n$.

Proof: Let T be any non trivial tree and $S = \{b_1, b_2, ..., b_n\}$ be the number of blocks of T with $|S| = n$. Since every edge of a tree T is a block in T. Clearly $q = n$ corresponds to a vertex in $L(T)$. Also by Theorem(2), $\gamma_{RCL}(G) \le q$. Therefore $\gamma_{RCL}(G) \le q = n$. Hence $\gamma_{RCL}(T) \le n$.

Theorem [9]: For any non trivial tree T, $\gamma_{RCL}(T) \le \gamma_{RC}(T)$.

Proof: Let $f = (V_0, V_1, V_2)$ be a γ_{RC}-function in $L(T)$ and $g = (V_0', V_1', V_2')$ be a γ_{RC}-function in T. Suppose $F = \{v_1, v_2, ..., v_n\}$ be the number of end vertices of T such that $D_c = V - F$ be the minimal γ_c-set of T. Then we have $\forall\ v \in V_1$, $V_0' = |F|$, $V_1' = |v|$ and $V_2' = |N(F)|$. Let $F' = \{e_1, e_2, \ldots, e_n\}$ be the γ_c'-set of T and $\{e_i; 1 \le i \le n\} \subseteq \{e_n\}$ be the number of edges incident with the vertices of $N(F) \cap D_c$ then $\exists$ the corresponding number of vertices $\{v_i\}$ of $L(T)$ such that $\{v_i\} \in V_2$. Further if for some edges of $\{e_i\}$ are incident with atleast two vertices of V_2'. Then $\exists$ $\{v_j\}, \{v_k\} \subseteq \{v_i\}$ such that $\{v_j\} \in V_2$, $\{v_k\} \in V_1$ and $F' - \{v_2\} \cup v_k \in V_1$, which gives $\gamma_{RCL}(T) \le \gamma_{RC}(T)$.

Theorem [10]: For any connected (p,q) graph G, $\gamma_{RCL}(G) + \gamma_c(\overline{G}) \le 2p + \gamma_c(G) - 2$.

Proof: We know that by Theorem [7], $\gamma_{RCL}(G) \le p + \gamma_c(G) - 1$ and by Theorem B, $\gamma_c(\overline{G}) \le p - 2$. Therefore $\gamma_{RCL}(G) + \gamma_c(\overline{G}) \le p + \gamma_c(G) - 1 + p - 2 = 2p + \gamma_c(G) - 3$. Hence $\gamma_{RCL}(G) + \gamma_c(\overline{G}) \le 2p + \gamma_c(G) - 3$.

Theorem [11]: For any connected (p,q) graph G, $\gamma_{RCL}(G) + \chi \le p + \gamma_c(G) + \Delta$.

Proof: By Theorem (7), $\gamma_{RCL}(G) \le p + \gamma_c(G) - 1$ and by Theorem [C], $\chi \le \Delta + 1$. Hence $\gamma_{RCL}(G) + \chi \le p + \gamma_c(G) + \Delta$.

Theorem [12]: For any graph G, $\gamma_{RCL}(G) + \kappa \le p + \gamma_c(G) + \delta - 1$ where κ denotes the connectivity of G.

Proof: The proof of this Theorem follows from Theorem (7) and Theorem[D].

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